Regular Languages and Nondeterministic Finite Automata

CMPU 240 • Language Theory and Computation • Spring 2019

Previously:

- Introduced finite automata as a model of computation

Today:

- Quiz
- Discuss the regular languages
- Introduce nondeterminism
- Assignment 1

Regular languages

Regular languages

DEFINITION. A language L is called a *regular language* if there exists a DFA D such that $L(D) = L$, i.e., there is a DFA that recognizes it.

Regular operations

Union ($\cup$):

$A \cup B = \{w \mid w \in A \text{ or } w \in B\}$

Concatenation ($\circ$):

$A \circ B = AB = \{xy \mid x \in A, y \in B\}$

Kleene star (*):

$A^* = \{w \mid w = x_1 x_2 \ldots x_k, k \geq 0, x_i \in A\}$

Example of regular operations

If $A = \{a, b\}$ and $B = \{b, c\}$, we get

$A \circ B = \{ab, ac, bb, bc\}$

Note for *, we stick together symbols in any way we want to get longer string

$A = \{a, b, c\}$

$A^* = \{\varepsilon, a, b, c, aa, bb, ab, ba, ac, bc, aba, abb, \ldots\}$

We get an ***infinite language*** unless $A \subseteq \{\varepsilon\}$

Note $\varepsilon \in A^*$, even if $\varepsilon \notin A$

Closure properties of regular languages

The collection of regular languages is *closed* under regular operations.

If we take 2 regular languages – or 1 regular language, for Kleene star – and apply a regular operation, we get another regular language.

"Closed" means "you can't get out" by using the operation.

Starting from a simple set of regular languages and applying these three operations, we can get to all regular languages.

Proof of closure under $\cup$

Show that if A and B are regular, then so is $A \cup B$.

Proof by construction:

Given the automata

$M_1 = (Q_1, \Sigma, \delta_1, q_1, F_1)$ recognizing A

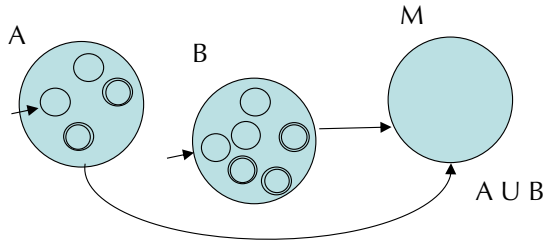
$M_2 = (Q_2, \Sigma, \delta_2, q_2, F_2)$ recognizing B

construct $M = (Q, \Sigma, \delta, q_0, F)$ recognizing $A \cup B$.

For simplicity, let the alphabets be the same.

You might think: Run the string through M_1, see whether M_1 accepts it, then run the string through M_2 and see whether M_2 accepts it.

But you can't try something on the whole input string, and try another thing on the whole input string; *you get only 1 pass!*



Approach: Simulate M_1 and M_2 simultaneously as we read each input symbol.

- Imagine putting two fingers on the diagrams of the automata for M_1 and M_2, and moving them around according to the input.
- At the end, if either finger is on an accept state, then we accept.

Implement this strategy in M.

Formalization

Keep track of a state in M_1 and a state in M_2 as a single state in M

- Each state in M corresponds to a pair of states, one in M_1 and one in M_2

Let $Q = Q_1 \times Q_2 = \{(q, r) \mid q \in Q_1, r \in Q_2\}$

How to define δ?

Define δ

When a new symbol comes in, go to wherever q goes and wherever r goes, individually

- $\delta((q, r), a) = (\delta_1(q, a), \delta_2(r, a))$

Start state is $q_0 = (q_1, q_2)$

Accept set is $F = (F_1 \times Q_2) \cup (Q_1 \times F_2)$

- Note $F_1 \times F_2$ would give the intersection, which isn't what we want.

It's clear by induction that the k^{th} state of M is just the k^{th} state of M_1 and k^{th} state of M_2.

Problem

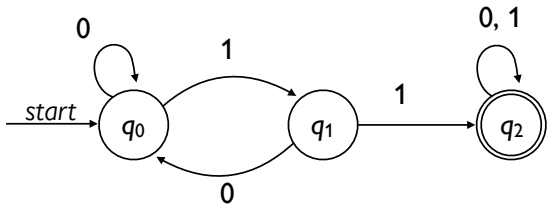
Prove that the collection of regular languages is closed under concatenation.

- But... where does the first string end and the second begin?
- As with union, the solution involves "keeping track of multiple possibilities"
- Next we'll consider a type of finite automaton that can keep track of multiple possibilities, much simplifying writing these proofs.

Nondeterministic finite automata

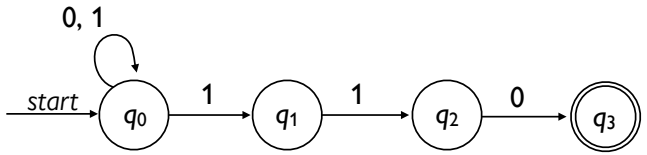
All of the FAs we have seen so far are *deterministic finite automata* (DFAs)

- Exactly *one possible move* from each state to another for a given input symbol.



Nondeterministic finite automata

Allow a finite automaton to have a choice of *zero or more* next states for each state–input pair.

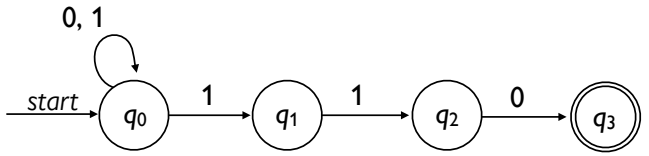


Note: two "1" arrows from q_0; can take either.

The present state does ***not*** determine the next state; there are multiple possible futures!

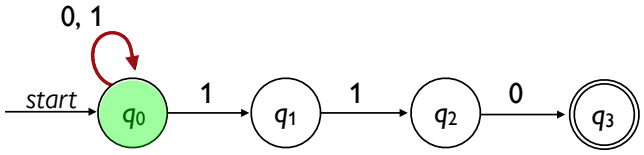
NFAs are structurally similar to DFAs, but they represent a fundamental shift in how we'll think about computation.

Example



0 1 0 1 1 0

Example



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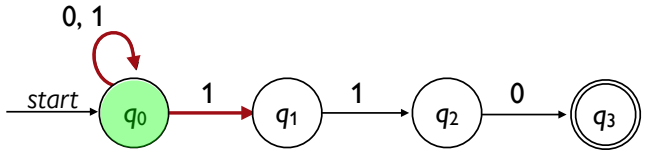
Begin at start state q_0.

Read **0**: Follow the loop back to q_0.

Example

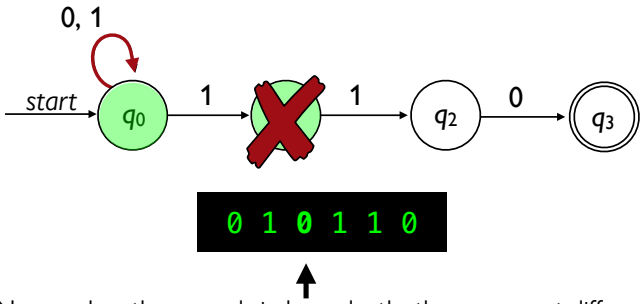


0 **1** 0 1 1 0

Read 1: There are 2 arrows labeled 1 starting at q_0.

Split into 2 paths to represent the 2 places machine could be: q_0 and q_1.

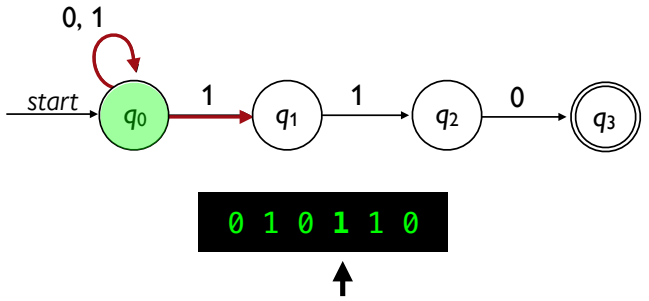
Example



Read 0: Now each path proceeds independently; they represent different threads of computation.

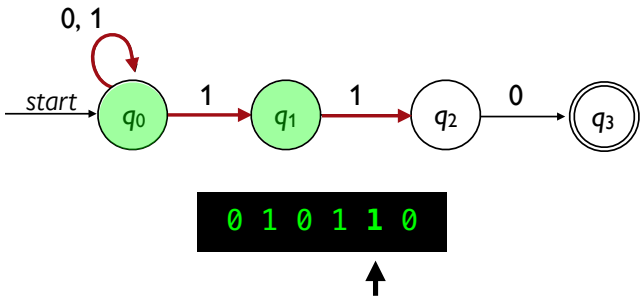
- The path at q_0 goes back to q_0.
- There's no place for the path at q_1 to go (no arrow with 0), so remove that path.

Example



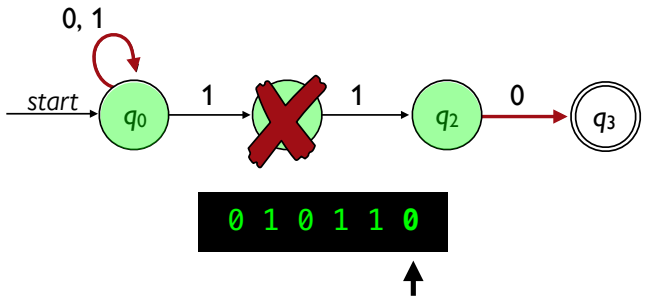
Read 1: Branch into q_0, q_1.

Example



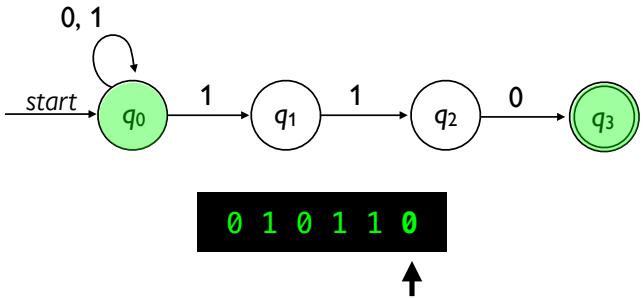
Read 1: Follow 1 arrows from q_0 and q_1 to get to q_0, q_1, q_2

Example

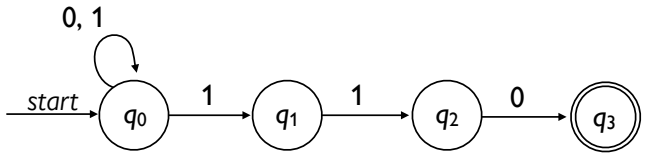


Read 0: Follow 0 arrows from q_0, q_1, q_2 to get to q_0, q_3

Example



The NFA *accepts* this string because at least one path ended up at an accepting state (q_3), i.e., `010110` $\in L(B)$



Each path represents a different thread of the computation.

By contrast, `01011` is not in $L(B)$ because paths end at q_0, q_1, q_2; *all possibilities are reject states*.

Nondeterministic machines are a serious departure from physical computers.

There are three ways to build an intuition for them:

- Perfect guessing
- Tree computation
- Massive parallelism

Perfect guessing

We can view nondeterministic machines as having magic superpowers that enable them to guess the correct choice of moves to make.

- If there is at least one choice leading to an accepting state for the input, the machine will guess it.
- If there are no choices, the machine guesses any one of the wrong answers.

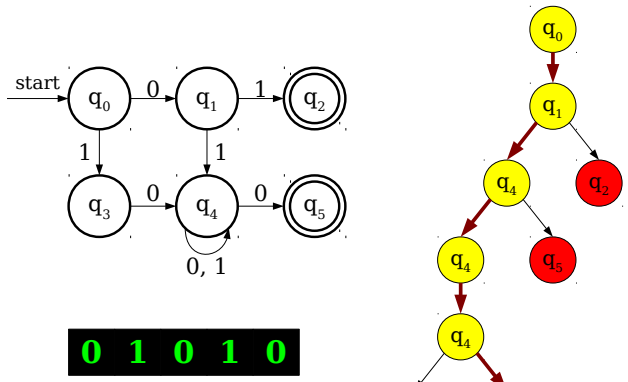
There's no physical analog for this style of computation.

NFA = Felix felicis

Tree computation

We can view nondeterminism as a tree, where at each decision point, the automaton clones itself for each possible decision.

- The series of choices forms a directed, rooted tree.
- At the end, if any active accepting states remain, we accept.

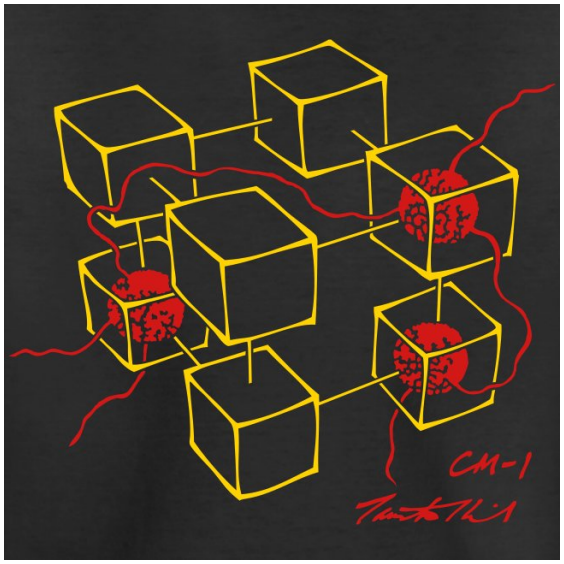


Massive parallelism

An NFA can be thought of as a DFA that can be in many states at once.

Each symbol read causes a transition on every active state into each potential state that could be visited.

Nondeterministic machines can be thought of as machines that can try any number of options in parallel.



We'll return to these intuitions to consider nondeterminism for different models through the semester.

Sometimes nondeterminism gives additional power; sometimes it doesn't.

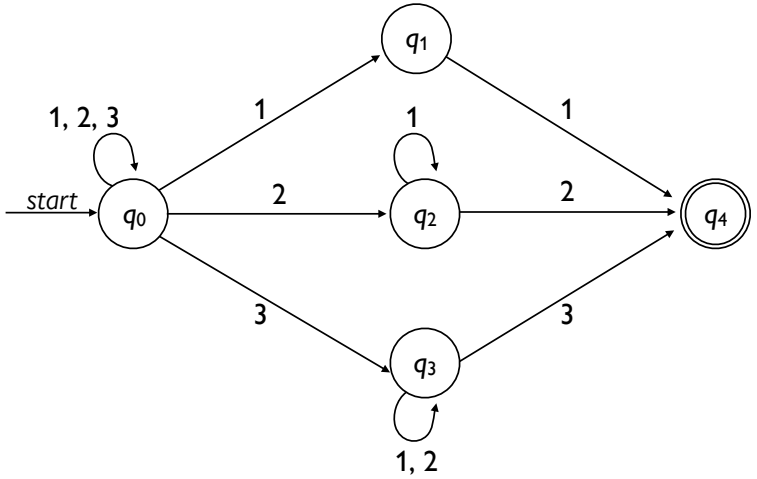
Problem

Design an NFA to accept strings over alphabet {1, 2, 3} such that the last symbol appears previously, without any intervening higher symbol, e.g.,

… 11
… 21112
… 312123

Trick: Use the start state to mean "I guess I haven't seen the symbol that matches the ending symbol yet".

Three other states represent a guess that the matching symbol has been seen, and remembers that symbol.



Formal definition of a nondeterministic finite automaton (NFA)

$N = (Q, \Sigma, \delta, q_0, F)$ like a DFA, but:

$\delta(q, a)$ is a *set of states*, rather than a single state

Extension to $\hat{\delta}$:

Basis: $\hat{\delta}(q, \varepsilon) = \{q\}$

Induction: Let
$\hat{\delta}(q, w) = \{p_1, p_2, \ldots, p_k\}$
$\delta(p_1, a) = S_i$ for $i = 1, 2, \ldots, k$
Then $\hat{\delta}(q, wa) = S_1 \cup S_1 \cup \cdots \cup S_k$

Set of states you can reach by starting in state q *and processing* w

Set of states you can reach by starting in state p_i *and processing* a

Set of states you can reach by starting in state q *and processing* wa

Language of an NFA: An NFA accepts *w* if *any* path from the start state to an accepting state is labeled *w*. Formally:

$L(N) = \{w \mid \hat{\delta}(q_0, w) \cap F \neq \varnothing\}$

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