Depth-First Traversal

Depth-First Traversal is another algorithm for traversing a graph.

Called depth-first because it travels "deeper" in the graph whenever possible.

Edges are explored out of the most recently discovered vertex v that still has unexplored edges. When all of v's edges have been explored, the search "backtracks" to explore the edges incident on the vertex from which v was discovered.

We will use an algorithm with a stack, S, to manage the set of nodes.

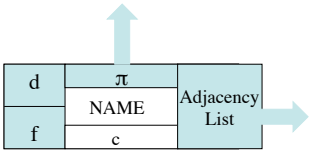
Depth-First Search

DFS algorithm maintains the following information for each vertex **u**:

- **u.c** (white, gray, or black) : indicates status
  white = not discovered yet
  gray = discovered, but not finished
  black = finished
- **u.d** : discovery time of node u
- **u.f** : finishing time of node u
- **u.π** : predecessor of u in Depth-First tree

DFS node

Each node has fields for predecessor (π), discovery time (d), finish time (f) and color (c). Each node also has an associated adjacency list with pointers to neighboring nodes.



Depth-First Search

```
DFS-Init (G, s):
 1. time = 0
 2. for all nodes v
 3.     v.d = v.f = ∞
 4.     v.c = white
 5.     v.π = NONE
 6. S = ∅
 7. DFS (G, s)
```

Initialize global timer to 0.

Set discovery time and finish times of all nodes to infinity and color them white.

Initialize stack S to ∅.

Call DFS (G, s)

Depth-First Search Using a Stack

```
DFS (G,s)
 1. S.push(s)
 2. while S is not empty
 3.     u = S.peek()
 4.     if u.c == WHITE
 5.         u.c = GRAY
 6.         u.d = time
 7.         time = time + 1
 8.         for all white neighbors v of u
 9             v.π = u
10.            S.push(v)
11.    else if u.c == GRAY
12.        S.pop()
13.        u.c = BLACK
14.        u.f = time
15.        time = time + 1
16.    else // u is BLACK
17.        S.pop()
18. end while
```

Complexity is based on number of edges |E|

Complexity (Adjacency List)
- check all edges adjacent to each node from both directions - O(E) time
- total = O(V + E) = O(V^2) (w.c.)

Depth-First Search (recursive version)

Initially, time (counter) = 0

After execution, for every vertex u, u.d < u.f

```
DFS (G)
1.  for each w ∈ G
2.      if w.c == white
3.          DFS-Visit (G,w)
```

Note: If G = (V, E) is not connected, then DFS will still visit the entire graph with the additional code above.

```
DFS-Visit (G,u)
1.  u.c = gray
2.  u.d = time
3.  time = time + 1
4.  for each v adjacent to u
5.      if v.c == white
6.          v.π = u
7.          DFS-Visit(G,v)
8.      end if
9.  end for
10. u.c = black
11. u.f = time
12. time = time + 1
```

```
public class DepthFirstSearch {
    private boolean[ ] marked;
    private int[ ] edgeTo;
    private int count = 0;

    public DepthFirstSearch(Graph G, int s) {
        marked = new boolean[|G.V()|];
        dfs(G, s);
    }
    private void dfs(Graph G, int v) {
        marked[v] = true;
        count++;
        for each w adjacent to v
            if (!marked[w])
                edgeTo[w] = v
                dfs(G,w);
    }
```

Enumerating shortest path, s→v

pathTo(v):

1. if (!marked[v]) return false
2. Stack<Integer> path = new Stack<Integer>()
3. for (int x = v; x != s; x = edgeTo[x])
4. path.push(x)
5. path.push(s)
6. return path

When pathTo finishes, the stack path will contain the dfs path from s to v and they can be popped off the stack in order.

Proposition B1: DFS marks all vertices connected to a given source in time proportional to the sum of their degrees.

Informal proof:

Every marked vertex is connected by a path to s since the algorithm finds vertices only by following edges. Now suppose that some unmarked vertex w is connected to s. Since s itself is marked, any path from s to w must have at least one edge from the set of marked to unmarked nodes. Since s is marked, any path from s to w must have at least one edge from the set of marked to unmarked vertices, say v-x. But the algorithm would have discovered x after marking v, so no such edge can exist, a contradiction. The time bound follows because marking ensures that each vertex is visited once (taking time prop to its degree to check marks

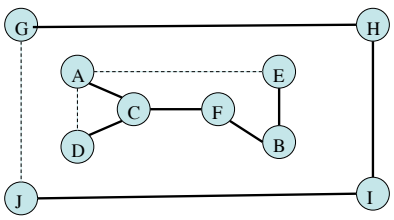
Analysis of Depth-First Search

Proposition B2: DFS allows us to provide clients with a path from a given source to any marked vertex in time proportional to the path length.

Informal proof:

By induction on the number of vertices visited, it follows that the the edgeTo array represents a tree rooted at the source. The pathTo method builds the path in time proportional to its length.

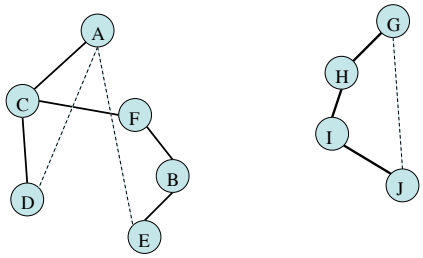
Example DFS Traversal



$d_{3,4}$ $e_{7,8}$ $b_{6,9}$ $f_{5,10}$ $c_{2,11}$ $a_{1,12}$ $j_{16,17}$ $i_{15,18}$ $h_{14,19}$ $g_{13,20}$

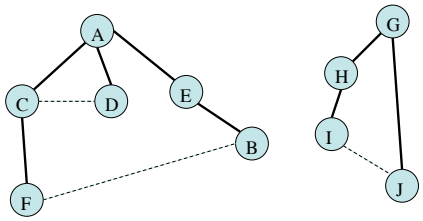
The first subscript indicates the time at which each node is discovered and pushed onto stack; the second indicates the time at which the node became a dead end.

Depth-first Search Forest



Tree edges are solid lines and dashed lines are back edges.

Breadth-first Search Forest



Tree edges are solid lines and dashed lines are cross edges.

DFS Tree

DFS builds a depth-first tree whose edges can be traced from any node to s using the π values at each node. Or by using the edgeTo[ ] array in the non-object-oriented version.

The DFS algorithm defines a depth-first forest G_π.

Topological Sort - Application of DFS

input: directed acyclic graph (DAG)
output: ordering of nodes s.t. if (u,v) ∈ E, then u comes before v in ordering

Topological-Sort (G)
1. call DFS(G,s) to compute finishing times v.f for each v
2. as each vertex is finished, insert it at *head* of a linked list
3. return the linked list of vertices

Complexity (Adjacency List Representation) - O(V + E)

Topologically sorted vertices are ordered in *reverse* order of their finishing times. An application of this type of sorting algorithm is to indicate precedence among ordered events represented in a DAG.

Classification of DFS Edges in a Directed Graph

1. **Tree edges:** Edges included in depth-first forest. Edge (u,v) is a tree edge if v was first discovered by edge (u,v).
2. **Back edges:** Edge (u,v) connects a vertex u to an ancestor (non-parent) v in a depth-first tree.
3. **Forward edges:** Edge (u,v) connects a vertex u to a descendant (non-child) v in a depth-first tree.
4. **Cross edges:** All other edges, i.e., between sibling nodes (e.g., nodes on different branches) of the same depth-first tree or between nodes in different depth-first trees.

Finding Strongly Connected Components of a Digraph

A digraph is strongly connected if, for any distinct pair of vertices u and v there exists a directed path from u to v and a directed path from v to u. In general, a digraph's vertices can be partitioned into disjoint maximal subsets of vertices that are mutually accessible via directed paths of the digraph; these subsets are called strongly connected components.

input: directed graph G
output: strongly connected components of G

1. do a DFS traversal of the digraph and number its vertices in the order that they become dead ends.
2. reverse the directions of all the edges of the digraph to get (G^T)
3. do a DFS traversal of G^T by starting the traversal at the highest numbered vertex and consider the vertices in order of decreasing *u.f*
4. output the vertices of each tree in the DFF from line 3 as G^{SCC}

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The strongly connected components G^{SCC} are exactly the subsets of vertices in each DFS tree obtained during step 3, the last traversal.

Time complexity of this algorithm?