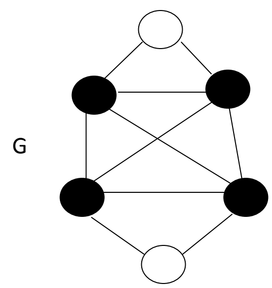
Clique Problem

Let G be an undirected graph. A *clique* of G is a subset V' of V such that each pair of u, v ∈ V' is connected by an edge.

CLIQUE:
instance: a graph G=(V, E) and a positive integer k ≤ V
question: is there a V' ⊆ V of size k such that each pair of vertices in V' is connected?



k = 4 (V' = the black nodes) is a yes instance for the clique problem on G.

Showing Clique is NPC

1. Show Clique ∈ NP.
   Given an instance (a graph) and a certificate (the set of vertices in V'), the validation requires checking each pair of vertices in V' to see if there is an edge between them. This requires $O(n^2)$ time.

2. Show Clique is NP-hard. (show 3-SAT $\leq_p$ Clique)
   Given an instance C of a 3 CNF formula (clauses and variables) and a clique certificate of size k, construct a graph G using positive integer k such that G has a Clique of size k iff C is satisfiable.

Showing Clique is NPC

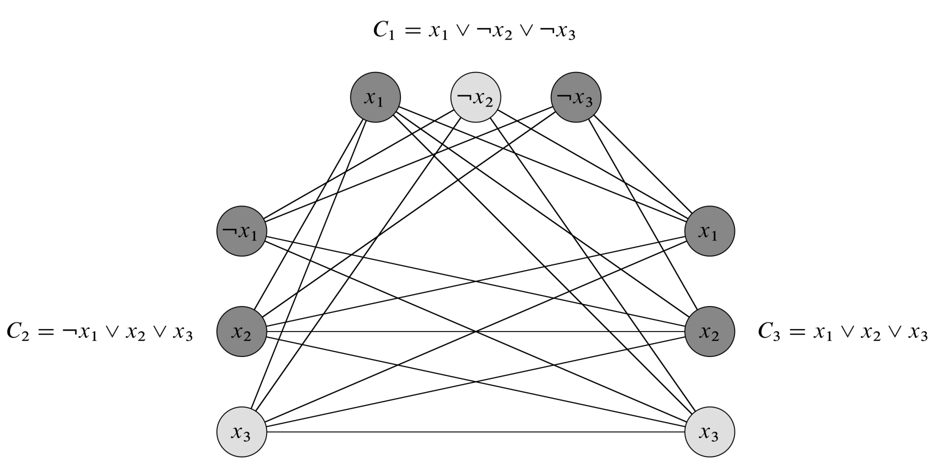
Given an instance C of a 3-SAT formula (clauses and variables), construct a graph G and choose a positive integer k such that G has a set CLIQUE of size k iff C is satisfiable.

For each clause of C, add a group of 3 vertices to G and add an edge between two vertices if

- **they are in different triples (they represent literals in different clauses) and**
- **if the literals are consistent (one is not the negation of the other).**

Choose k = the number of clauses in C.

The graph G can be constructed in time polynomial in the number of clauses in C.



Satisfying assignment has $x_2 = 0$, $x_3 = 1$, and x_1 = either 0 or 1.
This assignment satisfies C_1 with (not x_2), and C_2 and C_3 with x_3.

Showing Clique is NPC

Show if C $\in$ 3-SAT then G $\in$ Clique:

Suppose y is a satisfying truth assignment for C. Then each clause in C has at least one literal that is assigned 1 (true) when y is applied and each such literal corresponds to a vertex in a triple. Choosing one triple of connected vertices in G represents a true literal from each clause and yields a set V' of k vertices. To see that the set V' is a clique, note that each node in G representing a true literal is connected to the nodes representing true literals in other clauses because it can't be that the connected nodes are complements. So the transformation yields f(y), a k-clique on G.

Showing Clique is NPC

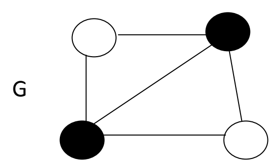
Show if G $\in$ Clique then C $\in$ 3-SAT:

Suppose f(y) is a clique V' of size k on G. No edges in G (constructed on last slide) connect vertices in the same triple and so V' contains exactly one vertex per triple. We can create a truth assignment y by assigning 1 to each literal in C that corresponds to a vertex in V' without the possibility of connecting a literal and its complement, since G contains no edges between inconsistent literals. Therefore, each clause is satisfied under y and C $\in$ 3-SAT.

Vertex Cover (VC)

Let G be an undirected graph. A *vertex cover* of G is a subset VC of V such that for every (u, v) ∈ E, at least one of u or v ∈ VC.

VC: Vertex Cover Problem
instance: a graph G=(V, E) and a positive integer k ≤ V
question: is there a subset V' ⊆ V of size k such that each edge in E has at least one endpoint in V'?



k = 2 (V' = the black nodes) is a yes instance of VC for this G.

Showing VC is NPC

1. Show VC ∈ NP.
   Given an instance and a certificate, the validation requires checking the ends of each edge to see if at least one end is in the vertex cover. In an n node graph, there are $O(n^2)$ edges, so this checking can be done in p-time in the number of edges.

2. Show VC is NP-hard. (show Clique $\leq_p$ VC)
   Given an instance G and positive integer k such that G has a clique V' ⊆ V of size k, take the complement of G (G_c) and show that G contains a clique of size k iff G_c contains a VC of size |V| - k.

Showing VC is NPC

Show if G $\in$ Clique then $G_c \in$ VC:
Suppose G has a clique V' with |V'| = k. Let (u,v) be any edge in G_c. Then (u,v) $\notin$ E, so u and v cannot both be part of the clique on G. Thus, at least one of u or v is in V - V', meaning that edge (u,v) is covered by V - V'. Since we picked edge (u,v) arbitrarily, the same must be true of every edge in G_c. Hence, the set V - V', which has size |V| - k, forms a vertex cover for G_c. So $G_c \in$ VC.

Show if $G_c \in$ VC then G $\in$ Clique:
Suppose G_c has a vertex cover V' of size |V| - k. This means that for all u, v $\in$ V, if u $\notin$ V' and v $\notin$ V', then (u,v) $\in$ E. So V - V' is a clique of size |V| - |V'| = k. Therefore G $\in$ Clique.