CMPU241 Analysis of Algorithms

Optimal Comparison-Based Sorting Algorithms

Sorting Algorithms (Ch. 6 - 8)

Slightly modified definition of the sorting problem:

input: A collection of n data items $<a_1,a_2,…,a_n>$ where data item a_i has a *key*, k_i, drawn from a linearly ordered set (e.g., ints, chars)

output: A permutation (reordering) $<a'_1,a'_2,…,a'_n>$ of the input sequence such that $k_1 \le k_2 \le … \le k_n$

- **In practice, one usually sorts objects according to their key (the non-key data is called *satellite data*.)**
- **If the records are large, we may sort an array of pointers based on some key associated with each record.**

Sorting Algorithms

- A sorting algorithm is *comparison-based* if the only operation we can perform on keys is to compare them.
- A sorting algorithm is *in place* if only a constant number of elements of the input array are ever stored outside the array.

Running Time of Comparison-Based Sorting Algorithms

	worst-case	**average-case**	**best-case**	**in place?**
Insertion Sort				
Merge Sort				
Heap Sort				
Quick Sort				

Heap-Sort (Chapter 6)

In order to understand heap-sort, you need to understand binary trees.

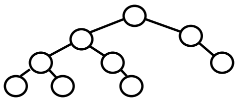
The algorithm doesn't use a data structure for nodes as you might be familiar with when working with binary trees.

Instead, it uses an array to abstract away from the complexity of linked binary trees. In so doing, the algorithm has a fast run time with low-cost operations: swapping the values in an array, like Insertion-Sort and Bubble-Sort do.

Binary Trees

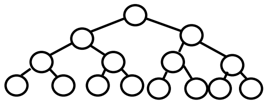
binary tree

- A rooted tree in which each internal node has at most 2 children



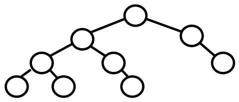
complete binary tree

- Each level of the binary tree is full.



Binary Trees

Some binary tree notation:



Node at top is **root** of tree.

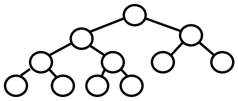
Nodes with no subtrees are **leaves** of tree.

Nodes with one or more subtrees are **internal nodes**.

The **depth** of a node v is the number of edges from node v to the tree's root node. A root node has a depth of 0.

The **height** of a node v is the number of edges on the *longest path* from node v to a leaf. A leaf node has a height of 0 and root has height of longest leaf to root path.

Heaps



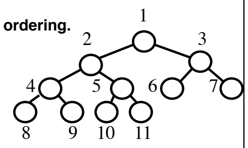
heap

- A heap is a complete binary tree or an almost-complete binary tree, with the requirement that it may be missing only the rightmost leaves on the bottom level.

  We say the bottom level is left-filled.

- Each node contains a key and the keys are some totally ordered, comparable type of data.

Heapsort



Imaginary nodes are numbered using **level ordering.**

A heap is represented with an array

- root is A[1]
- for element A[i]
  - left child is in position A[2i]
  - right child is in position A[2i + 1]
  - parent is in A[⌊i/2⌋]

Variables used for the array implementation of a heap

- ***heapsize*** is number of elements in heap
- ***length*** is number of positions in array

Note that current ***length*** of the array must be ≥ current ***heapsize***.

Max-Heap

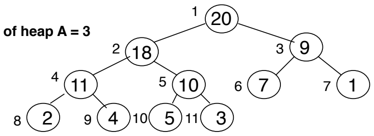
In the array representation of a max-heap, the root of the tree is in A[1]. Given an index i of a node,

Parent(i)	LeftChild(i)	RightChild(i)
return (⌊i/2⌋)	return (2i)	return (2i + 1)

Max-heap property: A[Parent(i)] ≥ A[i]

	1	2	3	4	5	6	7	8	9	10	11	: index
A	20	18	9	11	10	7	1	2	4	5	3	: keys

n = 11
height of heap A = 3



Max-heaps are used for, e.g., sorting data and for priority queues.

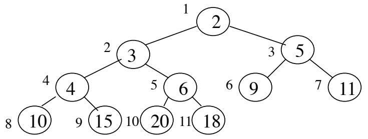
Min-Heap

In the array representation of a min-heap, the root of the tree is in A[1], and given the index i of a node,

Parent(i)	LeftChild(i)	RightChild(i)
return (⌊i/2⌋)	return (2i)	return (2i + 1)

Min-heap property: A[Parent(i)] ≤ A[i]

	1	2	3	4	5	6	7	8	9	10	11	: index
A	2	3	5	4	6	9	11	10	15	20	18	: keys



Min-heaps are used for, e.g., priority queues in event-driven simulators.

Creating a Heap: Build-Max-Heap

- Observation: Leaves are already trivial max-heaps.
  Elements A[(⌊n/2⌋ + 1) ... n] are leaves.
  Elements A[(1...⌊n/2⌋)] are internal nodes.
- Start at parents of leaves...then go to grandparents of leaves...moving larger values up the tree.

```
Build-Max-Heap(A)
for $i = \lfloor A.length/2 \rfloor$ downto 1
    Max-Heapify(A, i)
```

Running Time of Build-Max-Heap

- About *n/2* calls to Max-Heapify (O(n) calls)

Max-Heapify: Maintaining the Heap Property

- *Precondition*: when M-H is called on a node i, the subtrees rooted at the left and right children of A[i], A[2i] and A[2i + 1] are max-heaps (i.e., they obey the max-heap property)
- ...but subtree rooted at A[i] might not be a max-heap (that is, A[i] may be smaller than its left and/or right child)
- *Postcondition:* Max-Heapify will cause the value at A[i] to be compared and swapped with the largest child of A[i], to "float down" or "sink" in the heap until the subtree rooted at A[i] becomes a max-heap.
- In a totally unordered array, execution would start at the first parent node of a leaf because all leaves are max-heaps.

Max-Heapify: Maintaining the Max-Heap Property

Precondition: the subtrees rooted at 2i and 2i+1 are max-heaps when Max-Heapify(A, i) is called.

```
Max-Heapify(A, i)
1.  left = 2i          /* index of left child of A[i] */
2.  right = 2i + 1   /* index of right child of A[i] */
3.  largest = i
4.  if left <=  A.heap-size and A[left] > A[i]
5.      largest = left
6.  if right <=  A.heap-size and A[right] > A[largest]
7.      largest = right
8.  if  largest != i
9.       swap(A[i], A[largest])  /* swap i with larger child */
10.      Max-Heapify(A, largest) /* continue heapifying to the leaves */
```

Max-Heapify: Running Time

Running Time of Max-Heapify

- every line is $\theta(1)$ time except the recursive call in line 10.
- in worst-case, last level of binary tree is half empty and the sub-tree rooted at left child of root has size at most (2/3)*n*. Note that in a complete binary tree (CBT) the subtrees to left and right would be equal size.

We get the recurrence $T(n) <= T(2n/3) + \theta(1)$

which, by case 2 of the Master Theorem, has the solution

$$T(n) = \theta(\lg n)$$

Max-Heapify takes O(h) time when node A[i] has height h in the heap. The height h of a tree is the longest root to leaf path in the tree. h = O(lgn) in the worst case

Creating a Heap: Build-Max-Heap

- Observation: Leaves are already max-heaps.
  Elements $A[(\lfloor n/2 \rfloor + 1) \ldots n]$ are all leaves.
- Start at parents of leaves...then go up to grandparents of leaves...etc.

```
Build-Max-Heap(A)
A.heapsize = A.length
for i = ⌊A.length/2⌋ downto 1
    Max-Heapify(A, i)
```

Running Time of Build-Max-Heap

- About *n/2* calls to Max-Heapify (O(n) calls)

Correctness of Build-Max-Heap

- The entire array A meets the Max-Heap property.

Correctness of Build-Max-Heap

Loop invariant: At the start of each iteration *i* of the for loop, each node *i* + 1, *i* + 2, ..., *n* is the root of a max-heap.

- Initialization: *i* = ⌊*n/2*⌋. Each node ⌊*n/2*⌋ + 1, ⌊*n/2*⌋ + 2, ... *n* is a leaf, trivially satisfying the max-heap property.
- Inductive hypothesis: At the start of iteration k <= ⌊*n/2*⌋ and k >= 1, the subtrees of *k* are the roots of max-heaps
- Inductive step (maintenance): During iteration k, Max-Heapify is called on node k. By the IH, the left and right subtrees of k are max-heaps. When Max-Heapify is called on node k, the value in node k is repeatedly swapped with larger valued child until that value is greater than either child. Therefore value that was in node k is correctly positioned in the max-heap rooted at k.

```
Build-Max-Heap(A)
A.heapsize = A.length
for i = ⌊A.length/2⌋ downto 1
    Max-Heapify(A, i)
```

Correctness of Build-Max-Heap

Termination: at termination, *i = 0*. By the loop invariant, nodes *1, 2, ...,n* are the roots of max-heaps. Therefore the algorithm is correct.

```
Build-Max-Heap(A)
A.heapsize = A.length
for i = ⌊A.length/2⌋ downto 1
    Max-Heapify(A, i)
```

Reminder from last slide:

Loop invariant: At the start of each iteration *i* of the for loop, each *node i + 1, i + 2, ..., n* is the root of a max-heap.

Heap Sort

Input: An *n*-element array A (unsorted).
Output: An *n*-element array A in sorted order, smallest to largest.

```
HeapSort(A)
Build-Max-Heap(A)   /* rearrange elements to form max heap */
for i = A.length downto 2 do
    swap A[1] and A[i]       /* puts max in ith array position */
    A.heapSize =  A.heapSize - 1
    Max-Heapify(A,1)     /* restore heap property */
```

Relies on observation that the largest element in the array is at the top of the heap.

Does this algorithm have best case and worst case running times?

Heap Sort

Input: An *n*-element array A (unsorted).
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    A.heapSize =  A.heapSize - 1
    Max-Heapify(A,1)     /* restore heap property */
```

Build-Max-Heap(A) takes
= O(nlgn) time

Max-Heapify(A,1) takes
O(lg|A|) = O(lgn) time

Running time of HeapSort
- 1 call to Build-Max-Heap() ⇒ O(n) time
- n-1 calls to Max-Heapify() each takes O(lgn) time ⇒ O(nlgn) time

Heapsort Time and Space Usage

- An array implementation of a heap uses *O(n)* space, one array element for each node in heap.
- Heapsort uses *O(n)* space and is **in place**, meaning at most constant extra space beyond that taken by the input is needed.
- Running time is as good as merge sort, O(nlgn) in worst case.

Heaps as Priority Queues

Definition: A priority queue is a data structure for maintaining a set S of elements, each with an associated key. A max-priority-queue gives priority to keys with larger values and supports the following operations:

1. *insert(S, x)* inserts the element x into set S.
2. heap-*maximum(S)* returns value of element of S with largest key.
3. *extract-max(S)* removes and returns element of S with largest value key.
4. *increase-key(S,x,k)* increases the value of element x's key to new value k (assuming k is at least as large as current key's value).

Priority Queues: Application for Heaps

An application of max-priority queues is to schedule jobs on a shared processor. Need to be able to

check current job's priority...............Heap-Maximum(A)
remove job from the queue...............Heap-Extract-Max(A)
insert new jobs into queue................Max-Heap-Insert(A, key)
increase priority of jobs.....................Heap-Increase-Key(A,i,key)

Initialize PQ by running Build-Max-Heap on an array A.

A[1] holds the maximum value after this step.

Heap-Maximum(A) - returns value of A[1].
Heap-Extract-Max(A) - Saves A[1] and then, like Heap-Sort, puts item in A[heapsize] at A[1], decrements heapsize, and uses Max-Heapify(A, 1) to restore heap property.

Inserting Heap Elements

Inserting an element into a max-heap:

- increment heapsize and "add" new element to the highest numbered position of array
- go from new leaf to root, swapping values if child > parent. Insert input key at node in which a parent key larger than the input key is found

```
Max-Heap-Insert(A, key)
A.heapsize = A.heapsize +1
i = A.heapsize
while i > 1 and A[parent(i)] < key
    A[i] = A[parent(i)]
    i = parent(i)
A[i] = key
```

Here, values are moved up to where they should be in a max-heap.

Running time of Max-Heap-Insert: *O(lgn)*
- time to traverse leaf to root path (height = *O(lgn))*

Heap-Increase-Key

Heap-Increase-Key(A, i, key) - If key is larger than current key at A[i], moves node with increased key up heap until heap property is restored by exchanging it with its smaller parent until parent key is > A[i].

An application for a min-heap priority queue is an event-driven simulator, where the key is an integer representing the number of seconds (or other discrete time unit) from time zero (starting point for simulation).

Sorting Algorithms

- A sorting algorithm is *comparison-based* if the only operation we can perform on keys is to compare them.
- A sorting algorithm is *in place* if only a constant number of elements of the input array are ever stored outside the array.

Running Time of Comparison-Based Sorting Algorithms

	worst-case	**average-case**	**best-case**	**in place?**
Insertion Sort	n^2	n^2	n	yes
Merge Sort	$n\lg n$	$n\lg n$	$n\lg n$	no
Heap Sort	$n\lg n$	$n\lg n$	$n\lg n$	yes
Quick Sort				

Addendum

Build-Max-Heap – Tighter bound: O(n)

```
Build-Max-Heap(A)
1. A.heapsize = A.length
2. for i ← ⌊length(A)/2⌋ downto 1
3.       Max-Heapify(A, i)
```

Proof of tighter bound for O(n) relies on following theorem:

Theorem 1: The number of nodes at height h in a max-heap ≤ $\lceil n/2^{h+1} \rceil$

Height of a node v = largest number of edges from v to a leaf.
Depth of a node v = number of edges from node v to the root.

Tight analysis relies on the properties that an n-node heap has height at least floor of lgn and at most the ceiling of $n/2^{h+1}$ nodes at height h. The time for max-heapify to run at a node varies with the height of the node in the tree, and the heights of most nodes are small.

Lemma 1: The number of internal nodes in a proper binary tree is equal to the number of leaves in the tree - 1.

Defn: In a proper binary tree (pbt), each node has exactly 0 or 2 children.

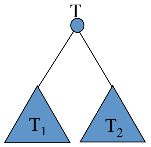
Let I be the number of internal nodes and let L be the number of leaves in a proper binary tree. The proof is by induction on the height of the tree.

Basis: h=0. I = 0 and L = 1. I = L - 1 = 1 - 1 = 0, so the lemma holds.

Inductive Step: Assume lemma is true for proper binary trees of height h (IHOP) and show for proper binary trees of height h + 1.

Consider the root of a proper binary tree T of height h+1. It has left and right subtrees (L and R) of height at most h.
$I_T = (I_L + I_R) + 1 = (L_L - 1) + (L_R - 1) + 1$ (by the IHOP) =
$(L_L + L_R - 2) + 1 = L_L + L_R - 1$. Since $L_T = L_L + L_R$ we have that $I_T = L_T - 1$.
QED

Diagramatic proof of Lemma 1



#Internal nodes in T = #Internal nodes in T_1 + #Internal nodes in T_2 + 1
= (#Leaves in T_1 - 1) + (#Leaves in T_2 - 1) + 1 (IHOP)
= (#Leaves in T_1 + #Leaves in T_2) - 2 + 1
= #Leaves in T - 1 (by observation that # of leaves in T is equal to # leaves in its subtrees.)

Theorem 1: The number of nodes at level h in a max-heap ≤ $\lceil n/2^{h+1} \rceil$

Let h be the height of the heap. Proof is by induction on h, the height of each node. The number of nodes in the heap is n.

Basis: Show the theorem holds for nodes with h = 0. The tree leaves (nodes at height 0) are at depths H and H-1.

Let x be the number of nodes at depth H, that is, the number of leaves assuming that n is a complete binary tree, i.e., that $n = 2^{h+1}-1$

Note that n - x is odd, because a complete binary tree has an odd number of nodes (1 less than a power of 2).

Theorem 1: The number of nodes at level h in a max-heap ≤ $\lceil n/2^{h+1} \rceil$

We have that n is odd and x is even, so all nodes have siblings (all internal nodes have 2 children.) By Lemma 1, the number of internal nodes = the number of leaves - 1.

So n = # of nodes = # of leaves + # internal nodes = 2(# of leaves) - 1. Thus, the #of leaves = (n+1)/2 = $\lceil n/2^{0+1} \rceil$ because n is odd.

Thus, the number of leaves = $\lceil n/2^{0+1} \rceil$ and the theorem holds for the base case.

Theorem 1: The number of nodes at level h in a max-heap ≤ $\lceil n/2^{h+1} \rceil$

Inductive step: Show that if thm 1 holds for height h-1, it holds for h.

Let n_h be the number of nodes at height h in the n-node tree T.

Consider the tree T′ formed by removing the leaves of T. It has $n' = n - n_0$ nodes. We know from the base case that $n_0 = \lceil n/2 \rceil$, so $n' = n - \lceil n/2 \rceil = \lfloor n/2 \rfloor$.

Note that the nodes at height h in T would be at height h-1 if the leaves of the tree were removed--i.e., they are at height h-1 in T′. Letting n'_{h-1} denote the number of nodes at height h-1 in T′, we have $n_h = n'_{h-1}$

$n_h = n'_{h-1} \le \lceil n'/2^h \rceil$ (by the IHOP) $= \lceil \lfloor n/2 \rfloor / 2^h \rceil \le \lceil (n/2)/2^h \rceil = \lceil n/2^{h+1} \rceil$.

Since the time of Max-Heapify when called on a node of height h is O(h), the time of B-M-H is

$$\sum_{h=0}^{\lg n} \frac{n}{2^{h+1}} O(h) = O(n \sum_{h=0}^{\lg n} \frac{h}{2^h})$$

and since the last summation turns out to be a constant, the running time is O(n).

Therefore, we can build a max-heap from an unordered array in linear time.