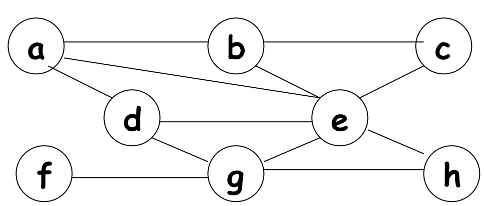
Graph Algorithms - Outline of Topics

- **Elementary Graph Algorithms** - Chapter 22
  - graph representation
  - breadth-first-search, depth-first-search, topological sort
- **Minimum Spanning Trees** - Chapter 23
  - Kruskal's and Prim's algorithms (greedy algorithms)
- **Single-Source Shortest Paths** - Chapter 24
  - Dijkstra's algorithm (greedy algorithm)

Undirected Graphs

An **undirected graph** *G = (V, E)* consists of

- A set *V* = |V| of nodes (vertices) , and
- A set *E* = |E| of undirected edges represented by node pairs



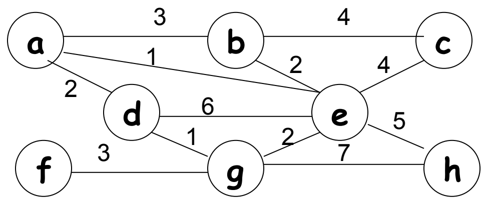
In this graph, both (b,c) and (c,b) are edges.

Graph Terminology

sparse graph: |E| is $o(|V|^2)$.

dense graph: |E| is $\Theta(|V|^2)$.

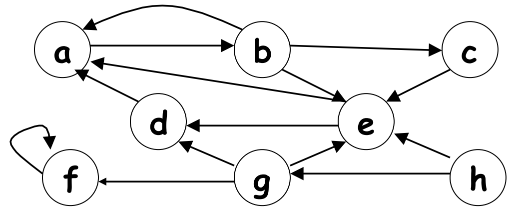
weighted graph: each edge has a number, called a *weight* attached to it. The weight is usually a positive number and may represent distance, cost, etc.



Graph Terminology

A **directed graph (digraph)** *G = (V, E)* consists of

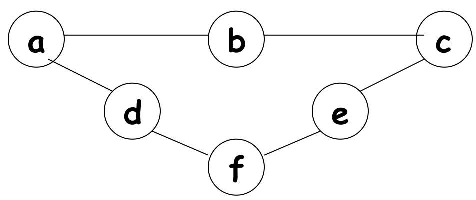
- A set *V* of nodes (vertices) , and
- A set *E* of *unidirectional* edges (represented by arrows)
- Self-loops are possible (as shown on node f)

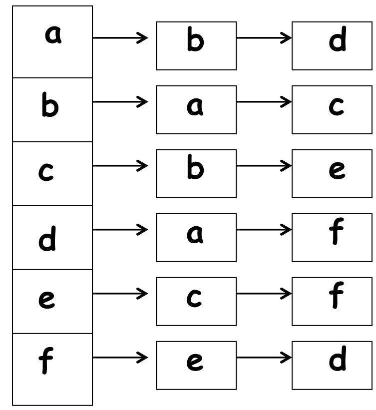


Note: In this graph, (b,c) is an edge, but (c,b) is not an edge.

Adjacency List Graph Representation

Adjacency list: An array A[1, |V|] of lists, one for each node v ∈ V (vertex set). Each node v's list contains pointers to all nodes adjacent to v in G. Each edge repeated twice.

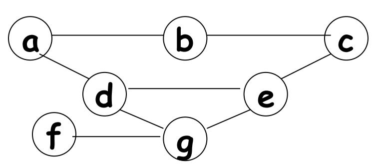




Complexity issues
- **advantage - storage is O(V + E) (good for sparse graphs)**
- **drawback - list traversal to find edge**

Representing Undirected Graphs with Adjacency Matrices

Adjacency matrix: An array *A*[*V*, *V*] such that
A[i,j] = 1 if (i,j) ∈ E and
0 otherwise

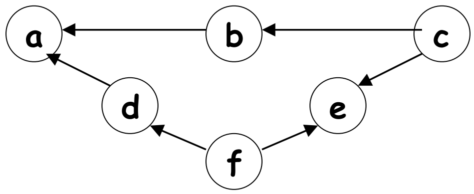


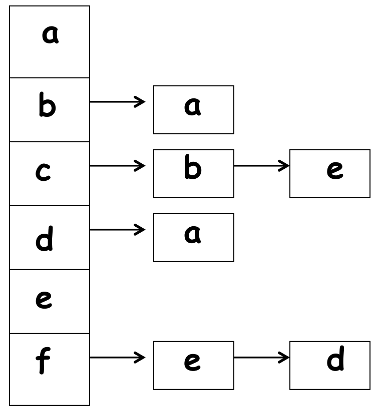
	a	b	c	d	e	f	g
a	0	1	0	1	0	0	0
b	1	0	1	0	0	0	0
c	0	1	0	0	1	0	0
d	1	0	0	0	1	0	1
e	0	0	1	1	0	0	1
f	0	0	0	0	0	0	1
g	0	0	0	1	1	1	0

Complexity issues
- advantage - O(1) time to check for edge
- drawback - storage is $O(V^2)$ (practical only for dense graphs)

In undirected graph, only the entries above the upper left to lower right diagonal need to be stored.

Representing Digraphs with Adjacency Lists





<u>Complexity issues</u>

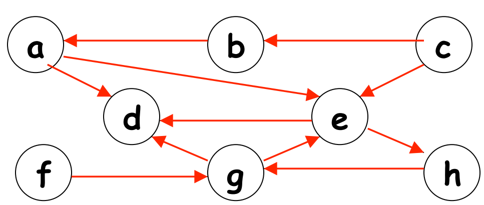
- **advantage - storage is O(V + E)**
  Good for sparse graphs, and most graphs we will use are sparse
- **drawback - list traversal to find edge**

Only store outgoing edges in adj. lists

If (u,v) is an edge, then it is **incident** on both u and v and we say vertex v is **adjacent** to vertex u. The same terms hold for undirected graphs. Adjacent vertices are called **neighbors**

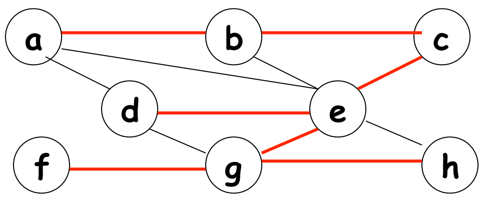
The **degree** of a node in an undirected graph is the number of edges incident on it.

The **in-degree** of a node in a digraph is the number of edges entering it and its **out-degree** is the number of edges leaving it.



A **path** of length k from a node u to a node u' is a sequence (v_0, v_1, ..., v_k) of nodes such that u = v_0, u' = v_k and there is an edge between each v_i, i = 0,1,2,...,k. In a digraph, a path exists between nodes a and b only if there is a sequence of *outgoing* edges from a to b

If there is a path p between vertices u and v, we say v is **reachable** from u via p.



A **simple** path has all distinct vertices.

The red edges in this graph trace a simple path between each pair of nodes.

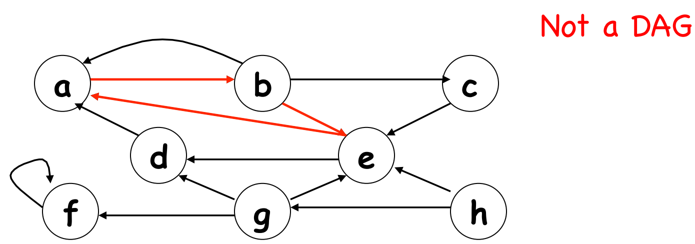
An undirected graph is **connected** if there is a simple path between every pair of nodes. A graph may have several **connected components** that are disjoint subsets of nodes.

A **completely connected** graph is an undirected graph in which every pair of nodes is adjacent.

In a digraph, a path (v_0, v_1, ..., v_k) forms a **cycle** if $v_0 = v_k$ and the path contains at least one edge

The cycle is **simple** if, in addition, v_1, v_2, ..., v_k are distinct.

A digraph with no cycles is called a *directed acyclic graph*, abbreviated DAG



Breadth-First Search

Breadth-First Search finds the shortest-path distance (number of edges) between a source node and every other node in G.

Called breadth-first because it discovers all vertices at distance k from a *source node* s before it discovers any vertices at distance k+1 from s, spanning the breadth before the depth of G.

BFS finds all vertices v that are reachable from s by building a breadth-first tree, where the path in the tree from s to v has the fewest number of edges of all paths from s to v.

Breadth-First Search

Breadth-First Search has time complexity of O(V + E) and is often used as a building block of other algorithms.

BFS is particularly useful in finding shortest paths on un-weighted graphs.

BFS starts at a node s in a graph and explores all its neighbor nodes before moving to the next level (neighbors of neighbors).

Explores nodes in "layers".

Maintains a queue of nodes to keep track of which node it should visit next.

Breadth-First Search Implementation

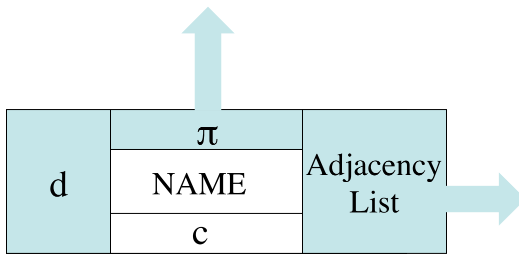
The algorithm from our book maintains a FIFO queue, Q, to manage the set of nodes and starts by enqueuing s, the source node

BFS algorithm maintains the following information for each vertex u:

- u.c: white, gray, or black to indicate status
  - white = not discovered yet; initially, all nodes except s are undiscovered.
  - gray = discovered, but not finished; initially only s.
  - black = finished; initially none are finished.
- u.d : distance from s to u; initially ∞ for all but s.d=0
- u.π : predecessor of u in BF tree; initially NIL for all (s.π = NIL and remains NIL)

BFS node

Each node has fields for predecessor (π), distance from source, and color. Each node also has an associated adjacency list with pointers to neighboring nodes.



Breadth-First Search

```
BFS (G, s):
s.c = gray; s.π = NIL
Q.enqueue (s) // Q is a FIFO ds
while Q ≠ ∅
    u = Q.dequeue()
    for each v adjacent to u
        if v.c == white
            v.c = gray
            v.d = u.d + 1
            v.π = u
            Q.enqueue(v)
    u.c = black
```

Q.enqueue(s) adds s to the rear of Q

Q.dequeue() removes and returns the item at the head of Q

Note: If G is not connected, then BFS will not visit the entire graph (without some extra provisions in the algorithm)

Breadth-First Search

```
BFS (G, s):
s.c = gray; s.π = NIL
Q.enqueue (s) // Q is a FIFO ds
while Q ≠ ∅
    u = Q.dequeue()
    for each v adjacent to u
        if v.c == white
            v.c = gray
            v.d = u.d + 1
            v.π = u
            Q.enqueue(v)
    u.c = black
```

Complexity (Adjacency List)

- each node enqueued and dequeued once = O(V) time
- each edge considered once (in each direction on undirected G) = O(E) time

• total = O(V + E)
  = $O(V^2)$ (w-c)

Analysis of Breadth-First Search

Shortest-path distance δ(s,v) : minimum number of edges in any path from vertex s to v. If no path exists from s to v, then δ(s,v) = ∞.

The ultimate goal of the proof of correctness is to show that v.d = δ(s,v) when the algorithm is done and that a path is found from s to all reachable vertices.

L. 22.1 : children of a node u are given a higher d value than u.
L. 22.2 : for every edge (u,v), the shortest path from s to v can be no longer than the (shortest path from s to u) + 1.
L. 22.3 : at any time, Q holds at most 2 distinct d values. I.e., the range of values in Q is at most 2. Why?
C. 22.4: the d values are monotonically increasing over time as the algorithm runs.

Theorem 22.5: (Correctness of BFS)
Let G = (V, E) be a directed or undirected graph, and suppose that BFS is run from a given source vertex $s \in V$. Then, during execution, BFS discovers every vertex $v \neq s$ that is reachable from the source s, and upon termination, $v.d = \delta(s,v)$ for every *reachable* or *unreachable* vertex v.

Proof by contradiction.

Assume that for some vertex v that $v.d \neq \delta(s,v)$ after running BFS. Also, assume that v is the vertex with *minimum* $\delta(s,v)$ that receives an incorrect d value. By Lemma 22.2, it must be that $v.d > \delta(s,v)$.
Case 1: v is not reachable from s. This is a contradiction to the assumption that v is reachable, and the Thm holds.
Case 2: v is reachable from s. Let u be the vertex immediately preceding v on a shortest path from s to v, so that $\delta(s,v) = \delta(s,u) + 1$. Because $\delta(s,u) < \delta(s,v)$ and because v is the vertex with the *minimum* $\delta(s,v)$ that receives an incorrect d value, $u.d = \delta(s,u)$.

So we have $v.d > \delta(s,v) = \delta(s,u) + 1 = u.d + 1$.

Consider the time t when u is dequeued. At time t, v is either white, gray, or black. We can derive a contradiction in each of these cases.

Case 1: v is white. Then in line 12, $v.d = u.d+1$.

Case 2: v is black. Then v was already dequeued, and therefore $v.d \le u.d$ (by L. 22.3).

Case 3: v is gray. Then v turned gray when it was visited from some vertex w, which was dequeued before u. Then $v.d = w.d + 1$. Since $w.d \le u.d$ (by L. 22.3), $v.d \le u.d + 1$.

Each of these cases is a contradiction to $v.d > \delta(s,v)$, so we conclude that $v.d = \delta(s,v)$. ■

Breadth-First Trees

BFS builds a breadth-first tree that can be identified by using the π values at each node.

The edges defined by each v.π are called tree edges.

```
Print-Path (G, s, v) // finds the tree edges between s and v,
if v == s            // starting at v
    print s
else
    if v.π == NIL
        print "no path from " s " to " v " exists"
    else
        Print-Path(G, s, v.π)
        print v
```

Breadth-First Search v2

```
BFS (G, s):
let marked be a boolean array of size |V| // init all false
let edgeTo be an array of |V| integers
Q.enqueue (s)
marked[s] = true
while Q ≠ ∅
    u = Q.dequeue()
    for each v adjacent to u
        if marked[v] == false
            edgeTo[v] = u
            marked[v] = true
            Q.enqueue(v)
```

Enumerating shortest path, s→v

```
pathTo(v):
 1. if (!marked[v]) return false
 2. Stack<Integer> path = new Stack<Integer>()
 3. for (int x = v; x != s; x = edgeTo[x])
 4.      path.push(x)
 5. path.push(s)
 6. return path
```

When pathTo finishes, the path will contain the path from s to v and they can be popped off the stack in order.

Analysis of Breadth-First Search

Proposition A1: For any vertex v reachable from s, BFS computes a shortest path from s to v such that no path from s to v has fewer edges.

Informal proof:

It is easy to prove by induction that Q always consists of zero or more vertices of distance k from s, followed by zero or more vertices of distance k+1 from s, for some integer k , starting with k = 0. This property implies, in particular, that vertices enter and leave Q in order of their increasing distance from s. When a vertex v enters Q, no shorter path to v will be found before v comes off Q, and no path to v that is discovered after v comes off Q can be shorter than the path length s to v.

Analysis of Breadth-First Search

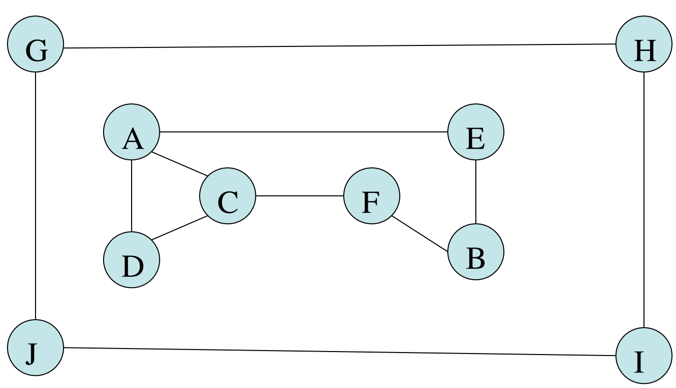
Proposition A2: BFS takes time proportional to V + E in the worst case.

Informal proof:

BFS marks all the vertices connected to s in time proportional to the degree of s. If the graph is connected, this sum equals the sum of the degrees of all the vertices, or 2E.

Initializing the marked[ ] and edgeTo[ ] arrays takes time proportional to V.

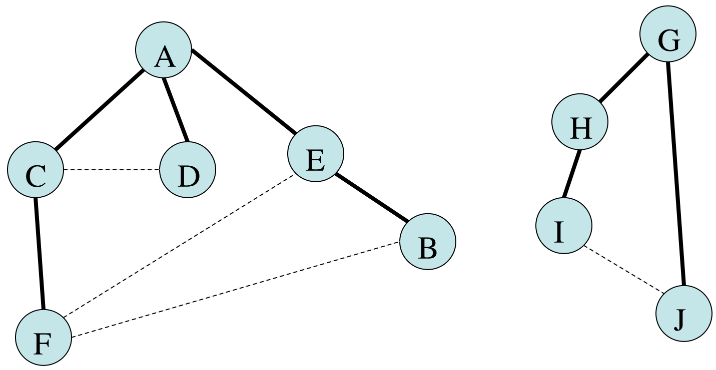
Example BFS Traversal



Order of visiting: a_1 c_2 d_3 e_4 f_5 b_6 g_7 h_8 i_9 j_{10}

Distance of vertex : 0 1 1 1 2 2 ∞ ∞ ∞ ∞

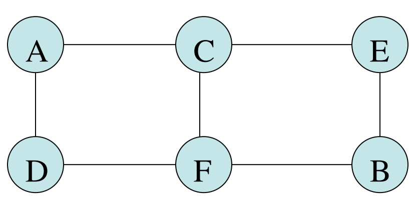
Breadth-first Search Forest



Tree edges are solid lines and dashed lines are cross edges.

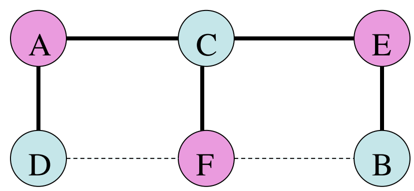
Bipartite Graphs

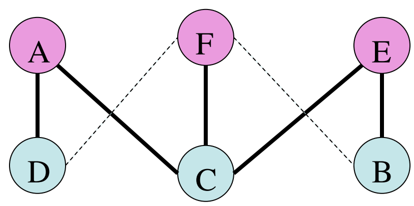
A graph is bipartite if all its vertices can be partitioned into two disjoint subsets X and Y so that every edge connects a vertex in X with a vertex in Y, i.e., if its vertices can be colored in 2 colors so that every edge has its end points colored in different colors.



Bipartite Graphs

Explain how BFS could be used to detect a bipartite graph.



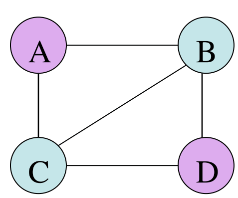


Mark the source, A, with color1, mark the nodes at level 1 with color2, and so on.

Every node on an even numbered level will be color1 and on every odd level color2

Bipartite Graphs

Is this graph bipartite?



Applications of BFS

Based upon the BFS, there are O(V + E)-time algorithms for the following problems:

- Testing whether graph is connected.
- Computing a spanning forest of graph.
- Computing, for every vertex in graph, a path with the minimum number of edges between start vertex and current vertex or reporting that no such path exists.
- Computing a cycle in graph or reporting that no such cycle exists.