Constraint-Satisfaction Problems

19 October 2020
Programming Assignment 1: Search

Programming Assignment 2: Multi-agent

Exam 1

Programming Assignment 3: Uncertainty

Out soon!

Programming Assignment 4: Classification

Exam 2

Final project / paper
The story so far
The story so far

Search and planning

- Define a state space
- Define a goal test
- Find path from start to goal
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- Define rewards, utility = (discounted) sum of rewards
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- Just like MDPs, but $T$ and $R$ are not known in advance
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Today: Constraint satisfaction
- Find solution that satisfies constraints
- Not just for finding a sequential plan
Constraint-satisfaction problems
What’s search for?

**Planning**: sequences of actions

Want: *path* to the goal

Paths have various costs and depths

Heuristics give problem-solving guidance
What’s search for?

**Planning**: sequences of actions

Want: *path* to the goal
Paths have various costs and depths
Heuristics give problem-solving guidance

**Identification**: assignments to variables

The *goal* itself is important, not the path
All paths at the same depth (for some formulations)
CSPs are a specialized class of identification problems
Constraint-satisfaction problems

Standard search problems:

State is a “black box” – any data structure
Goal test can be any function over states
Successor function can also be anything
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Constraint satisfaction problems (CSPs):

State is defined by variables $X_i$ with values from a domain $D$
Goal test is a set of constraints specifying allowable combinations of values for subsets of variables
Constraint-satisfaction problems

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Simple example of a formal representation language
Constraint-satisfaction problems

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Simple example of a formal representation language

Allows useful general-purpose algorithms with more power than standard search algorithms
Spectrum of representations

Atomic

Factored

Structured

Search, game-playing

CSPs, planning, propositional logic, Bayes nets, neural nets

First-order logic, databases, probabilistic programs
CSP examples
Example: Map coloring

**Variables:** WA, NT, Q, NSW, V, SA, T

**Domains:** \( D = \{ \text{red, green, blue} \} \)

**Constraints:** Adjacent regions must have different colors

- **Implicit:** WA \( \neq \) NT
- **Explicit:** (WA, NT) \( \in \) \{ (red, green), (red, blue), \ldots \}
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*Implicit:* WA \( \neq \) NT

*Explicit:* \((WA, NT) \in \{(\text{red, green}), (\text{red, blue}), \ldots\}\)*

**Solutions** are assignments satisfying all constraints, e.g.,

\{WA = \text{red}, NT = \text{green}, Q = \text{red}, NSW = \text{green}, V = \text{red}, SA = \text{blue}, T = \text{green}\}
**Constraint graphs**

*Binary CSP*: Each constraint relates (at most) two variables.

*Constraint graph*: Nodes are variables; arcs show constraints.

General-purpose CSP algorithms can use the graph structure to speed up search.

E.g., Tasmania is an independent subproblem!
Example: $n$-queens
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Formulation 1:

Variables: $X_{ij}$

Domains: \{0, 1\}

Constraints:
Example: $n$-queens

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$$\forall i, j, k \ (X_{ij}, X_{ik}) \in \{(0, 0), (0, 1), (1, 0)\}$$
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\[
\forall i, j, k \ (X_{ij}, X_{k:j}) \in \{(0, 0), (0, 1), (1, 0)\}
\]

\[
\forall i, j, k \ (X_{ij}, X_{i+k,j+k}) \in \{(0, 0), (0, 1), (1, 0)\}
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  \]
  \[
  \forall i, j, k \quad (X_{ij}, X_{kj}) \in \{(0, 0), (0, 1), (1, 0)\}
  \]
  \[
  \forall i, j, k \quad (X_{ij}, X_{i+k,j+k}) \in \{(0, 0), (0, 1), (1, 0)\}
  \]
  \[
  \forall i, j, k \quad (X_{ij}, X_{i+k,j-k}) \in \{(0, 0), (0, 1), (1, 0)\}
  \]
  \[
  \sum_{i,j} X_{ij} = N
  \]
Example: $n$-queens

Formulation 2:

Variables: $Q_k$

Domains: {1, 2, 3, ..., $N$}

Constraints:

*Implicit*: $\forall i, j \text{ non-threatening}(Q_i, Q_j)$

*Explicit*: $(Q_1, Q_2) \in \{(1, 3), (1, 4), \ldots\}$

...
Example: Sudoku

**Variables:**

Each (open) square

**Domains:**

\{1, 2, ..., 9\}

**Constraints:**
Example: Sudoku

Variables:
Each (open) square

Domains:
\{1, 2, \ldots, 9\}

Constraints:
9-way alldiff for each column
Example: Sudoku

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9-way alldiff for each row
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- 9-way alldiff for each column
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- 9-way alldiff for each region
Example: Sudoku

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Each (open) square

Domains:
\{1, 2, \ldots, 9\}

Constraints:
9-way alldiff for each column
9-way alldiff for each row
9-way alldiff for each region
(or can have a bunch of pairwise inequality constraints)
Example: The Waltz algorithm

The Waltz algorithm is for interpreting line drawings of solid polyhedra as 3D objects.

An early example of an AI computation posed as a CSP

*Approach:*

- Each intersection is a variable
- Adjacent intersections impose constraints on each other
- Solutions are physically realizable 3D interpretations
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Varieties of CSPs and constraints
Varieties of CSPs

Discrete variables

Finite domains
- Size $d$ means $O(d^n)$ complete assignments
- E.g., Boolean CSPs, including Boolean satisfiability (NP-complete)

Infinite domains (integers, strings, etc.)
- E.g., job scheduling, variables are start/end times for each job
- Linear constraints solvable
- Nonlinear constraints undecidable
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Linear constraints solvable

Nonlinear constraints undecidable

Continuous variables

E.g., start/end times for Hubble Telescope observations

Linear constraints solvable in polynomial time by linear-programming methods
Varieties of constraints

*Unary constraints* involve a single variable (equivalent to reducing domains), e.g.,

\[ SA \neq \text{green} \]

*Binary constraints* involve pairs of variables, e.g.,

\[ SA \neq WA \]

*Higher-order constraints* involve three or more variables, e.g.,

Sudoku row constraints
Real-world CSPs

Scheduling problems, e.g., which class is offered when and where?

Hardware configuration / circuit layout

Transportation scheduling

Factory scheduling

Fault diagnosis

… and many more!

Many real-world problems involve real-valued variables…
Solving CSPs
Standard search formulation

States defined by the values assigned so far (partial assignments)

- **Initial state**: the empty assignment, `{}`
- **Successor function**: assign a value to an unassigned variable
- **Goal test**: the current assignment is complete and satisfies all constraints

We’ll start with the straightforward, naïve approach, then improve it.
Search methods

What would BFS do?

Level 1  

Level 2  

...  

Level $N$  

solution!
Search methods

What would BFS do?

What would DFS do?

Let’s see!
Demo: Naïve (depth-first) search
Search methods

What would BFS do?

What would DFS do?

Let’s see!

What problems does naïve search have?
Backtracking search
Backtracking search is the basic uninformed algorithm for solving CSPs. It’s depth-first search with two improvements:
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*Idea 1*: One variable at a time

Variable assignments are commutative, e.g.,

\[
[\text{WA} = \text{red} \text{ then } \text{NT} = \text{green}] \text{ same as } [\text{NT} = \text{green} \text{ then } \text{WA} = \text{red}]
\]

so fix the ordering to get a smaller branching factor.

Only need to consider assignments to a single variable at each step
**Backtracking search** is the basic uninformed algorithm for solving CSPs. It’s depth-first search with two improvements:

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**Idea 2:** Check constraints as you go

Only consider values that don’t conflict with previous assignments.

 Might have to do some computation to check the constraints.

“Incremental goal test”
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**Idea 2:** Check constraints as you go

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“Incremental goal test”

Can solve $n$-queens for $n \approx 25$
Backtracking example
Backtracking example
Backtracking example
Backtracking example
Backtracking example
**Demo:** Backtracking
Improving backtracking
Improving backtracking

General-purpose ideas give huge gains in speed
Improving backtracking

General-purpose ideas give huge gains in speed

*Filtering:*

  Can we detect inevitable failure early?

*Ordering:*
Improving backtracking

General-purpose ideas give huge gains in speed

*Filtering:*
Can we detect inevitable failure early?

*Ordering:*
Which variable should be assigned next?
In what order should its values be tried?
Filtering
**Filtering**: Keep track of domains for unassigned variables and cross off bad options

**Forward checking**: Cross off values that violate a constraint when added to the existing assignment
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Demo: Backtracking with forward checking
Ordering
**Variable ordering**: *Minimum remaining values* (MRV):

Choose the variable with the fewest legal values left in its domain

*Why min rather than max?*

Also called “most constrained variable”

“Fail-fast” ordering
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Value ordering: Least-constraining value

Given a choice of variable, choose the least constraining value
I.e., the one that rules out the fewest values in the remaining variables
Note that it may take some computation to determine this! (E.g., rerunning filtering)

Why least rather than most?

Combining these ordering ideas makes 1000 queens feasible
Demo: Ordering
Iterative improvement
Iterative algorithms for CSPs

*Local search* methods typically work with “complete” states, i.e., all variables assigned.
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To apply to CSPs:

- Take an assignment with unsatisfied constraints
- Operators *reassign* variable values
- No tree, no frontier!
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Algorithm: While not solved,

*Variable selection:* randomly select any conflicted variable

*Value selection:* *min-conflicts* heuristic:

Choose a value that violates the fewest constraints
Example: 4-queens

**States:** 4 queens in 4 columns ($4^4 = 256$ states)

**Operators:** move queen in column

**Goal test:** no attacks

**Evaluation:** $h(n) = \text{number of attacks}$
Demo: Iterative improvement – coloring
Summary

CSPs are a special kind of search problem:

- States are partial assignments
- Goal test defined by constraints

Basic solution: backtracking search

Speed-ups:

- Ordering
- Filtering

Iterative min-conflicts is often effective in practice
Local search
Local search

Tree search keeps unexplored alternatives on the frontier (ensures completeness)

*Local search*: Improve a single option until you can’t make it better

*New successor function*: local changes

Generally much faster and more memory efficient (but incomplete and suboptimal)
Hill climbing

Simple, general idea:

Start wherever.
Repeat: Move to the best neighboring state.
If no neighbors better than current, quit.
Hill climbing quiz

Starting from X, where do you end up?
Starting from Y, where do you end up?
Starting from Z, where do you end up?
Simulated annealing

**Idea:** Escape local maxima by allowing downhill moves

But make them rarer as time goes on

```plaintext
function SIMULATED-ANNEALING(problem, schedule) returns a solution state
  inputs: problem, a problem
           schedule, a mapping from time to “temperature”
  local variables: current, a node
                   next, a node
                   T, a “temperature” controlling prob. of downward steps
  current ← MAKE-NODE(INITIAL-STATE[problem])
  for t ← 1 to ∞ do
    T ← schedule[t]
    if T = 0 then return current
    next ← a randomly selected successor of current
    ΔE ← VALUE[next] - VALUE[current]
    if ΔE > 0 then current ← next
    else current ← next only with probability e^{ΔE/T}
```
Genetic algorithms use a natural selection metaphor

Keep best $N$ hypotheses at each step (selection) based on a fitness function

Also have pairwise crossover operators, with optional mutation to give variety

Possibly the most misunderstood, misapplied (and even maligned) technique in AI
Example: $n$-queens

Why does crossover make sense here?

When wouldn’t it make sense?

What would mutation be?

What would a good fitness function be?
Acknowledgments

The lecture incorporates material from:

Pieter Abbeel, Dan Klein, et al., University of California, Berkeley;
ai.berkeley.edu
Ketrina Yim (illustrations)